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STABILITY OF IONIZED BEAMS

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Based on a solution of a system of the Poisson–Vlasov equations a dispersion equation is obtained for a wide class of initial disturbances taking into account the influence of boundary conditions on development of the instability in ionized beams. A procedure for determining the stability limit for both the unrestricted and restricted multicomponent ionized beams is suggested.

Introduction. The instability of ionized beams is the process of buildup of small disturbances with time relative to their equilibrium state [1]. However, this problem has not been studied to the fullest extent, especially concerning simulation of the process of beam propagation in real systems [2, 3]. In this aspect, investigations of characteristics of the development of the instability in multicomponent restricted beams are of scientific and practical significance. For instance, in probing the earth's ionosphere by a quasineutral beam it is important to know the conditions of instability development in the ion beams of inert gases (argon, xenon, etc.). In simulation of the process of beam propagation in power and technological plants it is desirable to have information about the influence of the boundary conditions and dimensions of a system on the stability of this process.

In the present work, by the instability of ionized beams is understood the process of their destruction caused by a deviation of the velocity distribution function of particles from its equilibrium value (Maxwellian). In solving the problem on the instability of plasma beams it is possible to analyze only the electrostatic instability, assuming that the disturbances of an electric field are potential. This assumption is justified by the fact that under the action of the electrostatic instability the equilibrium state of a plasma changes before manifestation of the magnetic effects since, as analysis shows, the increments of the electrostatic instability are substantially larger than those of the electromagnetic instability [1–4]. Development of the electrostatic instability leads to fluctuations of the electrostatic potential, thus causing a scatter of the charged particles by inhomogeneities of the potential which can be considered as an increase in the frequency of electron-ion collisions.

Dispersion Equation. We will consider a nonstationary system representing a multicomponent plasma with relative motion of its components. According to the collisionless model the small disturbances of such a system satisfy the linearized Poisson–Vlasov system [1]:

$$\frac{\partial}{\partial t} f_j(t, v, x) + v\frac{\partial}{\partial x} f_j(t, v, x) = \frac{e_j}{m_j}\frac{\partial f_{0j}}{\partial v}\frac{\partial \varphi(t, x)}{\partial x}, \quad \frac{\partial^2}{\partial x^2}\varphi(t, x) = -4\pi\sum_{j=1}^{N} e_j \int_{-\infty}^{\infty} f_j(t, v, x)\, dv\,. \tag{1}$$

Applying to system (1) Laplace transformation with respect to time t and Fourier transformation with respect to the coordinate x at the zero boundary conditions, we arrive at the following expression for the Fourier transform of the electrostatic potential φ:

$$\xi^2\varphi(\omega, \xi) = -4\pi\sum_{j=1}^{N}\int_{-\infty}^{\infty}\frac{1}{v}\left[\frac{e_j^2}{m_j}\frac{\partial f_{0j}(v)}{\partial v}\, i\xi\varphi(\omega, \xi) + f_{0j}(0, v, \xi)\right]\frac{idv}{\xi + \dfrac{\omega}{v}}\,. \tag{2}$$

If in Eq. (2) it is assumed that $\varphi(x) = \exp(ikx)$ or if the electrostatic potential $\varphi(x)$ is considered to be a smooth function that sufficiently rapidly decreases at infinity, i.e., such that the Fourier transformation exists for it in the classical sense, then from Eq. (2) the following dispersion relation stems for an unrestricted system [2]:

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$$\varepsilon(k,\omega) = \sum_{j=1}^{N} \frac{4\pi n_j e_j^2}{m_j k^2} \int_{-\infty}^{\infty} \frac{k}{\omega - kv} \frac{\partial}{\partial v} f_{0j}(v)\, dv + 1\,. \tag{3}$$

Let us consider a restricted system with the boundary conditions

$$\varphi(\omega, 0) = \varphi(\omega, l) = 0\,. \tag{4}$$

Any locally integrable function can be represented by the Fourier series

$$\varphi(x) = \sum_{n=-\infty}^{n=\infty} \varphi(n) \exp\left(\frac{i\pi n}{l} x\right). \tag{5}$$

In this case, a Fourier transformation for $\varphi(x)$ is as follows:

$$\varphi(\xi) = \sum_{-\infty}^{\infty} \varphi(n)\, \delta\left(\xi + \frac{\pi n}{l}\right). \tag{6}$$

By virtue of boundary condition (4), expression (6) is transformed as

$$\varphi(\xi) = \sum_{n=1}^{\infty} \varphi(n) \left[\delta\left(\xi + \frac{\pi n}{l}\right) - \delta\left(\xi - \frac{\pi n}{l}\right)\right]. \tag{7}$$

Substituting (7) into (2) and multiplying the scalars of both sides of (2) by $\left[\delta\left(\xi + \frac{\pi n}{l}\right) - \delta\left(\xi - \frac{\pi n}{l}\right)\right]$, we obtain the following expression for the Fourier transform of a disturbance of the electrostatic potential:

$$\varphi_n = \frac{i}{2} \sum \left\{ \int_{-\infty}^{\infty} \left[\frac{f_{0j}\left(0, v, \frac{\pi n}{l}\right)}{\omega + \frac{\pi n}{l} v} - \frac{f_{0j}\left(0, v, \frac{\pi n}{l}\right)}{\omega - \frac{\pi n}{l} v} \right] dv \right\} \left[\left(\frac{\pi n}{l}\right)^2 - 2\pi \sum \frac{e_j^2}{m_j} \int_{-\infty}^{\infty} \frac{v \left(\frac{\pi n}{l}\right)^2}{\left(\frac{\pi n v}{l}\right)^2 - \omega^2} \frac{\partial f_{0j}}{\partial v}\, dv \right]^{-1}. \tag{8}$$

The problem on development of oscillations in a plasma reduces to determination of the denominator zeros on the right-hand side of expression (8). Equating the denominator of the latter to zero, we obtain the dispersion equation

$$1 = 2\pi \sum_j \frac{e_j^2}{m_j} \int_{-\infty}^{\infty} \frac{v}{\left(\frac{\pi n v}{l}\right)^2 - \omega^2} \frac{\partial f_{0j}(v)}{\partial v}\, dv\,, \tag{9}$$

which in a linear approximation generalizes a wide class of the disturbances in ionized beams with account for the boundary conditions and allows investigation of their stability.

Of practical interest is an ionized beam consisting of the ions of inert gases (argon or xenon) compensated by electrons that propagates in a plasma medium (for instance, in the earth's ionosphere a heavy component of which is an oxygen ion). In this case, the problem on the stability of an ionized beam, in which the concentrations of electrons and ions much exceed the concentrations of these components in the ionosphere, reduces to determination of the electron-ion instability. A similar problem emerges when an electric current runs in the ionosphere.

Unrestricted Multicomponent Beam. We will consider a one-dimensional model for an unrestricted multicomponent plasma beam. The dynamics of the electrostatic small-amplitude oscillations in such a system is determined by a system of the Poisson–Vlasov equations [1] from which in a linear approximation dispersion equation (3) follows; it can be written in the form [2, 5]

$$\varepsilon (k, \omega) = 1 + \sum_{j=1}^{n} \frac{\omega_{pj}}{k^2} \int_{-\infty}^{\infty} \frac{k}{\omega - kv} \frac{\partial f_{0j}}{\partial v} dv , \tag{10}$$

where $\omega_{pj} = (4\pi n_j e_j^2 / m_j)^{1/2}$ is the plasma frequency of the jth component; for electrons, $\omega_{p.e} = 2\pi 10^4 \sqrt{n_e}$ Hz.

Mathematical formulation of the problem on the stability of solutions of the system of equations (1) implies determination of such solutions of dispersion equation (10) for which the imaginary part of the complex frequency is $\text{Im}\, \omega > 0$. In this case, in a linear approximation a solution of the system of the equations will be unstable since the oscillation amplitude of small disturbances increases with time without limit. As analysis shows, the limit of the beam stability is determined by the critical velocity of charged particles u_0, which, in turn, depends on a number of the thermophysical parameters such as the temperature, the concentration, the mass of the charged particles, and so on. By the critical velocity is understood such a directed velocity of the charged particles of a plasma u_0 that for any other relative velocity of the particles $v < u_0$ no plasma instability develops. Thus, the problem on determination of the limit of the beam instability of the plasma reduces to determination of the critical velocity.

Let us consider the case where the initial velocity distribution of particles f_0 is Maxwellian:

$$f_{j0} = \frac{1}{\sqrt{\pi}} \frac{n_j}{v_{T_j}} \exp \left[- \frac{(v - v_j)^2}{v_{T_j}^2} \right] ,$$

where $v_{T_j} = \sqrt{2\kappa T_j / m_j}$ is the thermal velocity of the jth component. With account for this relation, dispersion equation (10) can be written for the distribution function as

$$k^2 + \sum_{j=1}^{n} d_j^{-2} G (Z_j) = 0 , \tag{11}$$

where

$$d_j^2 = \frac{\kappa T_j}{4\pi n_j e_j^2} ; \quad Z_j = \frac{1}{v_{T_j}} \left(\frac{\omega}{k} - v_j \right); \quad G (Z) = 1 + Z \exp (- Z^2) \left[i \sqrt{\pi} - 2 \int_0^Z \exp (u^2)\, du \right] .$$

First, we will analyze the instability of a two-component plasma. For this case, dispersion equation (11) is written in the form [2]

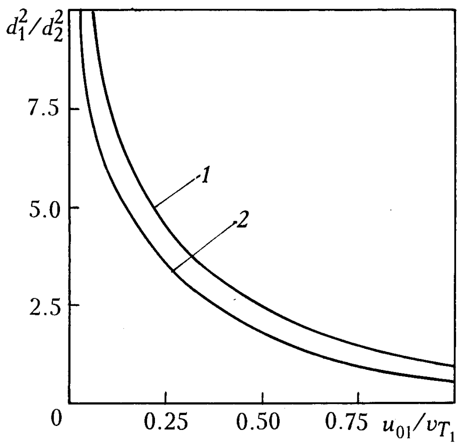
$$k^2 + d_1^{-2} G (Z_1) + d_2^{-2} G (Z_2) = 0 . \tag{12}$$

We will assume that the directed velocity of the second component is equal to zero. This is achieved by introducing the coordinate system related to the motion of this component. Then

$$Z_1 = \frac{1}{v_{T_1}} \left(\frac{\omega}{k} - v_1 \right), \quad Z_2 = \frac{1}{v_{T_2}} \frac{\omega}{k} . \tag{13}$$

A solution of dispersion equation (12) is determined by the points of intersection of the curves

$$\xi_1 = - (kd_1)^2 - G (Z_1) , \tag{14}$$



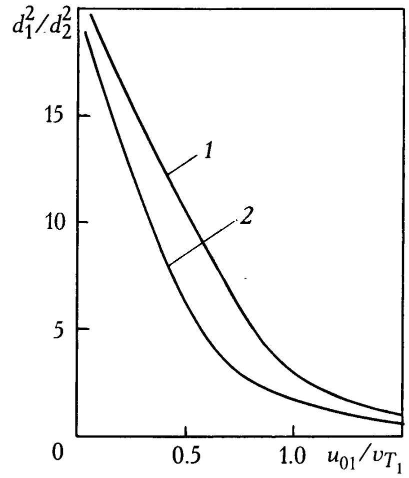


Fig. 1. Ratio of the critical u_{01} to thermal v_{T_1} velocities of electrons for the electron-ion instability of the plasma containing ions of oxygen, xenon, and argon (1), and hydrogen (2).

Fig. 2. Ratio of the critical u_{01} to thermal v_{T_1} velocities of ions for the ion-ion instability of the plasma containing the ions of argon (1) and xenon (2).

$$\xi_2 = \left(\frac{d_1}{d_2}\right)^2 G\,(Z_2)\,. \tag{15}$$

We consider k to be a real number; then the circular velocity ω is a complex velocity. If the imaginary part of the complex frequency is $\operatorname{Im}\omega > 0$, then the amplitude of oscillations increases with time, thus leading to instability of the system. Consequently, the boundary of the region of instability is determined from the condition $\operatorname{Im}\omega = 0$ or, which is the same, by solving Eq. (12) at $\operatorname{Im} Z_1 = 0$ $(j = 1, 2)$, which corresponds to intersection of curves (14) and (15) given above at real values of the argument.

It is pertinent to note that for sufficiently large k the curve corresponding to dependence (14) shifts far to the left along the real axis $\operatorname{Re} G(Z)$ so that it does not intersect with curve (15) at $\operatorname{Im} Z_1 > 0$, i.e., only damped modes $\operatorname{Im}\omega > 0$ will take place in the plasma. Consequently, there exists a value of k_0 dependent on d_1 and d_2 such that any wave of length $\lambda < 2\pi/k_0$ damps.

The critical value of the velocity u_{01} determining the limit of instability depends on k. We will consider the case $k = 0$ (an infinitely long wave) since to precisely this value of k a minimum critical velocity corresponds. Otherwise, any wave in a plasma will damp if the directed velocity is $v_1 < u_{01}$, where u_{01} is the critical velocity determined at $k = 0$. For the fixed relation d_1^2/d_2^2 at $k = 0$, we determine graphically the points Z_1^0 and Z_2^0 and from relations (13) obtain the following expression for the critical velocity:

$$\frac{u_{01}}{v_{T_1}} = -\,Z_1^0 + \frac{v_{T_2}}{v_{T_1}}\,Z_2^0\,. \tag{16}$$

For the case of the electron-ion instability of the plasma containing oxygen, argon, and xenon, Fig. 1 provides the values of the critical velocity of electrons u_{01} based on their thermal velocity v_{T_1}. For comparison, the corresponding critical velocities for hydrogen ions are also given. From the figure it follows that an increase in the mass of the ionic component leads to a decrease in the critical velocity of the electrons. This difference is most pronounced at the condition $1 < d_1^2/d_2^2 < 10$. The relation d_1^2/d_2^2 at the equal concentrations of the components is equal to the ratio of the electron and ion temperatures T_1/T_2. If the directed velocity of electrons is commensurable to their thermal velocity, this case corresponds to the limit of instability. If the electron temperature is two or threefold higher than the temperature, then at $v_1 = v_{T_1}$ the instability of the plasma develops. Thus, for $T_1/T_2 = 2$, when electrons move relative to oxygen ions, we have $u_{01}/v_{T_1} = 0.521$. It should be noted that in the presence of a current in the beam, $v_1 >> v_{T_1}$ as a

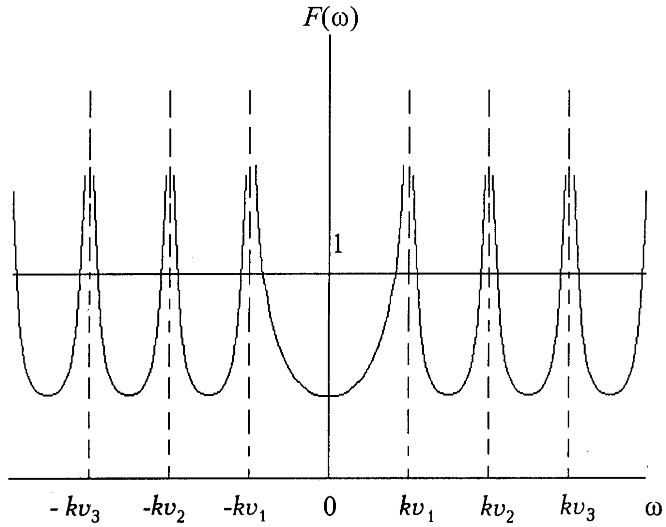


Fig. 3. Illustration of changing the function $F(\omega)$ for the multicomponent plasma.

rule, and the beam current is, consequently, unstable. Figure 2 gives data for the ion-ion instability [5]. In this case, in formula (16) by v_{T_1} is understood the thermal velocity of oxygen ions while by u_{01}, their directed velocity. The dependence shown in Fig. 1 is similar to the dependence for the electron-ion instability (Fig. 2). However, the thermal velocity of the electrons is 171 times higher than the thermal velocity of the oxygen ions at the same temperature. Consequently, in the case of the ion-ion instability, the absolute value of the critical velocity of the oxygen ions relative to xenon or argon ions is 171 times smaller than that of the critical velocity of the electrons in the case of the electron-ion instability. This means that the ion-ion instability develops even if the electron-ion instability does not exist.

Thus, the problem on determination of the critical velocity for the four-component plasma consisting of two components of ions and two components of electrons, where the ions and electrons move with the same velocity through the plasma at rest, reduces to the problem on determination of the critical velocity for the ion components. For instance, for the motion of xenon ions with a concentration of $10^{13}\ \mathrm{m}^{-3}$ and a temperature of 0.3 eV relative to the oxygen ions with a concentration of $10^{12}\ \mathrm{m}^{-3}$ and a temperature of 0.15 eV the critical velocity is u_{01}/v_{T_1} = 0.471.

Restricted Multicomponent Beam. We will investigate the influence of boundary conditions on the instability of plasma oscillations within the framework of formal hydrodynamics, i.e., we will assume that there is no heat scattering and the initial distribution functions are the Dirac δ-functions [6].

Integrating the right-hand side of Eq. (9) and employing the definition of the derivative of the δ-function, we arrive at the dispersion relation

$$2\pi \sum_j \omega_{pj}^2 \frac{\omega^2 + (kv_j)^2}{[(kv_j)^2 - \omega^2]^2} = 1\,. \tag{17}$$

In the case of a restricted system, periodic oscillations (Re $\omega = 0$) are of particular interest. Let us consider the left-hand side of Eq. (17) as a function of $F(\omega)$ (Fig. 3). The condition of the presence of complex solutions of Eq. (17) is equivalent to the case where all minima of the function $F(\omega)$ lie higher than unity. Decomposing each term on the left-hand side of Eq. (17) according to the equality

$$\omega_{pj}^2 \frac{\omega^2 + (kv)^2}{[(kv)^2 - \omega^2]^2} = \frac{1}{2}\frac{\omega_{pj}^2}{(\omega - kv)^2} + \frac{1}{2}\frac{\omega_{pj}^2}{(\omega + kv)^2}\,,$$

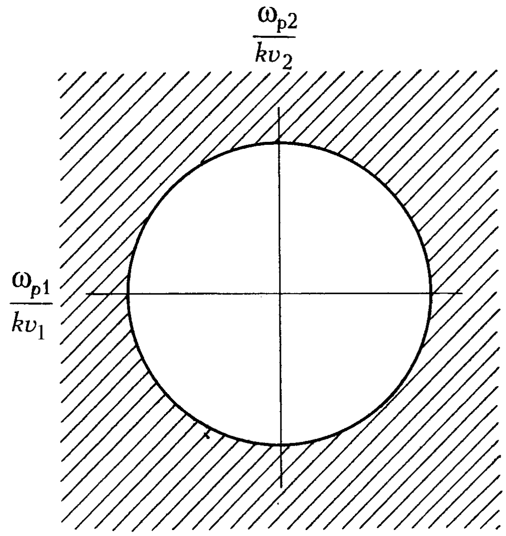


Fig. 4. Region of instability for the two-component confined plasma.

we obtain the dispersion equation for an unconfined plasma. Each component moving in the same direction disintegrates formally into two opposite waves moving with the same absolute velocity and plasma frequencies that are $\sqrt{2}$ times smaller than the initial one.

It is pertinent to note that in the vicinity of Re $\omega = 0$ a qualitative difference between the restricted and unrestricted beams — an unstable branch of the oscillations appears that has not existed earlier in the unrestricted system — is observed, which is attributable to the boundary conditions. The condition of buildup of these oscillations is presented by the inequality

$$\sum_{j=1}^{N} \frac{\omega_{pj}^2}{(kv_j)^2} > 1 . \tag{18}$$

From this relation it follows that the longest wave is the most unstable one and hence for the restricted system $k = \pi n/l$ ($n = 1, 2, ...$), the condition of instability acquires the form

$$\sum_{j=1}^{N} \frac{\omega_{pj}^2}{\left(\frac{\pi n}{l} v_j\right)^2} > 1 , \tag{19}$$

that allows simple geometric interpretation. If, in the N-dimensional space, we plot the complex ω_{pj}/kv_j on each axis, then the external region of the N-dimensional sphere with unit radius is the region of instability. The case of the two-flow instability is depicted in Fig. 4.

Let us consider several particular cases.

1. For a restricted electron beam moving relative to immobile infinitely heavy ions, the condition of instability acquires, according to (19), the form $\omega_{p.e}^2 > (\pi n v_1/l)^2$, where v_1 is the directed velocity of the electron beam.

2. Let two identical plasma components move in opposition with the same velocity v_1. In this case, the condition of instability development $\omega_{p1}^2 > 0.5(\pi n v_1/l)^2$ coincides with the corresponding condition for an unrestricted system of counterbeams only with the difference that in the latter instead of the value of $\pi n/l$ an arbitrary wave number k is used. From comparison of the instability conditions of the restricted and unrestricted systems, it follows that if the function $F(\omega)$ of the unrestricted system is even and $v_j \neq 0$, then the conditions of instability development in the vicinity of $\omega = 0$ in the unrestricted system are similar to those in the restricted system (Fig. 3).

3. The region of two-flow instability corresponds to the hatched region in Fig. 4. In this case, the condition of instability is represented by the inequality

$$\left(\omega_{p1} \Big/ \frac{\pi n}{l} v_1\right)^2 + \left(\omega_{p2} \Big/ \frac{\pi n}{l} v_2\right)^2 > 1 \, .$$

If, in this expression, one of the plasma frequencies tends to zero, the condition of instability corresponds to the Pierce problem [2]. From the dispersion equation for an unrestricted system (10) it follows that when one of the plasma frequencies tends to zero, only real solutions of the equation $F(\omega, k) = 1$ are possible for two unrestricted flows, since one of the two asymptotes disappears. In a restricted system (unlike the unrestricted ones), complex-conjugate solutions of the equation $F(\omega, k) = 1$ can exist.

4. In practice, the problem on motion of a restricted quasineutral beam in an ionospheric plasma is interesting. According to expression (19), the condition of instability of such a beam is the inequality

$$\left(\frac{\omega_{p.e1}}{\frac{\pi n}{l} v_1}\right)^2 + \left(\frac{\omega_{p.e2}}{\frac{\pi n}{l} v_2}\right)^2 + \left(\frac{\omega_{p.i1}}{\frac{\pi n}{l} v_1}\right)^2 + \left(\frac{\omega_{p.i2}}{\frac{\pi n}{l} v_2}\right)^2 > 1 \, ,$$

where $\omega_{p.e1}$ and $\omega_{p.e2}$ are the plasma frequencies of electrons of the beam and the ionosphere; $\omega_{p.i1}$ and $\omega_{p.i2}$ are the plasma frequencies of ions of the beam and the ionosphere; v_1 and v_2 are the directed velocities of ions of the beam and of the ionospheric plasma relative to the beam boundaries, respectively. In some applications, the case is important where the boundaries of the beam move in the ionosphere; then by l must be meant the distance between the boundaries.

Conclusions. A procedure of determination of the conditions of development of the instability of ionized multicomponent beams is developed that allows for the influence of such factors on the instability development as the ratio of the temperatures and concentrations of the components, their directed velocities, and dimensions of the system. It is shown that in restricted ionized beams the spectrum of wave numbers narrows from continuous to discrete and a new branch of the possible instability appears. The suggested calculated model of determination of the boundary of stability of ionized beams is in fair agreement with the experimental studies [6].

NOTATION

d, Debye radius; *e*, particle charge; *f*, disturbance of the velocity distribution function of particles relative to its stationary homogeneous background; f_0, initial value of the velocity distribution function of particles; *k*, wave number; *l*, beam length; *m*, particle mass; *n*, natural number; n_j, number of particles of the species *j* per unit volume; *T*, temperature; *t*, time; u_0, critical velocity of particles; v_j, directed velocity of particles along the coordinate *x*; v_T, thermal velocity of particles; *x*, coordinate directed along the beam; $\delta(x)$, delta-function; ε, dielectric constant; κ, Boltzmann constant; ξ, variable in the Fourier transformation; φ, disturbance of the electrostatic potential; ω, circular frequency; ω_p, plasma frequency of a particle. Subscripts: *j*, number of the component (particles); e, electron parameters; i, ion parameters; 0, initial value; p, particle.

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